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# Free-field realization of $D$ -dimensional cylindrical gravitational waves

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## Abstract

We find two-dimensional free-field variables for  $D$ -dimensional general relativity on spacetimes with  $D-2$  commuting spacelike Killing vector fields and non-compact spatial sections for  $D > 4$ . We show that there is a canonical transformation which maps the corresponding two-dimensional dilaton gravity theory into a two-dimensional diffeomorphism invariant theory of the free-field variables. We also show that the space-time metric components can be expressed as asymptotic series in negative powers of the dilaton, with coefficients which can be determined in terms of the free fields.

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# 1 Introduction

Symmetry reductions of higher-dimensional gravity theories are interesting for several reasons. One is that new classical solutions and new integrable models can be obtained. Furthermore, these solutions can be also interpreted via dimensional reduction as solutions for  $D = 4$  general relativity with matter. Second, the quantization of such systems can be helpful for understanding conceptual problems of quantum gravity [1, 2, 3], as well as for understanding issues related with quantum properties of black holes [4]. And third, new mathematical structures arise.

Particularly interesting are symmetry reductions which give  $D = 2$  integrable models, because all the relevant issues can be more easily explored. Free-field realizations of  $D = 2$  integrable models are extremely useful for understanding the structure of the space of solutions and especially for the quantization of the theory. One can obtain them via Backlund transformations, as in the case of the Liouville model [5], or via free-field realizations of the symmetry algebras [6]. In the context of  $D = 2$  integrable models of gravity, free-field realizations have been found for many models [7, 8, 9, 10] and this has been explained in [10] as the consequence of the integrability and special properties of the  $D = 2$  diffeomorphism algebra of the constraints.

In this paper we will consider  $D$ -dimensional general relativity, with  $D > 4$ , on spacetimes with  $D - 2$  commuting spacelike Killing vector fields. In  $D = 4$ , this system corresponds to cylindrically symmetric general relativity. The corresponding dynamical systems are exactly integrable, and the integrability for  $D > 4$  follows trivially from the proof of integrability in the  $D = 4$  case in the Belinski-Zakharov-Maison (BZM) approach [11, 12]. One simply replaces the relevant two by two matrices with  $D - 2 \times D - 2$  matrices. The symmetry reduced theory is a  $D = 2$  dilaton gravity coupled to a coset space sigma model [13]. In the  $D = 4$  case when the spatial section is non-compact, which corresponds to cylindrical gravitational waves, it has been shown in [10] that there is a canonical transformation which maps the constraints of the symmetry reduced theory into a free-field form, so that the initial theory is equivalent to a two-dimensional diffeomorphism invariant theory of four free fields. Since  $D > 4$  case involves only bigger matrices, it is reasonable to expect that the free-field construction of [10] could be also generalized, so that the  $D > 4$  reduced theory should be equivalent to a two-dimensional diffeomorphism invariant theory of  $1 + \frac{1}{2}(D - 1)(D - 2)$  free fields.

# 2 Two-dimensional dilaton gravity formulation

$D$ -dimensional spacetimes with  $D - 2$  commuting Killing vector fields are described by a line-element of the form

$$ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu + g_{ab}(x)d\chi^a d\chi^b , \quad (1)$$

where  $x^\mu$  are the two-dimensional coordinates and  $\partial/\partial\chi^a$  are the Killing vectors [14]. Let  $D = n+2$ , and we split the matrix  $g_{ab}$  into the determinant and the  $SL(n, R)$  piece

$$g_{ab}(x) = \varphi^{2/n}(x)\Lambda_{ab}(x) \quad (2)$$

so that  $\sqrt{\det g} = \varphi(x)$  and  $\det \Lambda = 1$ . The corresponding Einstein equations can be derived from the following two-dimensional action [13]

$$S = \int d^2x \sqrt{-g}\varphi \left[ R - \frac{1}{4} \text{tr}(\Lambda^{-1} \nabla^\mu \Lambda \Lambda^{-1} \nabla_\mu \Lambda) \right], \quad (3)$$

where  $g = \det g_{\mu\nu}$ ,  $R$  is a two-dimensional scalar curvature and  $\nabla_\mu$  are covariant derivatives. This action describes a two-dimensional dilaton gravity coupled to  $SL(n, R)/SO(n)$  coset space  $\sigma$ -model.

One can describe different physical solutions by appropriate choices of the Killing vectors and the spatial topology. The gravitational waves correspond to the case when all Killing vectors are spacelike and the spatial section is noncompact. When the spatial section is compact, one has the cosmological models. Axisymmetric stationary solutions correspond to the case when one of the Killing vectors is timelike.

The standard approach to study the dynamics of the action (3) is to fix completely the two-dimensional diffeomorphism invariance, so that the complete dynamics is contained in the Ernst equation for the matrix  $\Lambda$

$$\nabla_\mu(\varphi \Lambda^{-1} \nabla^\mu \Lambda) = 0. \quad (4)$$

For cylindrical waves, this is done in  $\varphi = r$  gauge, where  $x^\mu = (t, r)$ . Note that for  $n = 2$ , the eq. (4) has a duality symmetry. If

$$\Lambda = \frac{1}{\Delta} \begin{pmatrix} h^2 + \Delta^2 & h \\ h & 1 \end{pmatrix} \quad (5)$$

is a solution then

$$\tilde{\Lambda} = \frac{\Delta}{r} \begin{pmatrix} \tilde{h}^2 + \frac{r^2}{\Delta^2} & \tilde{h} \\ \tilde{h} & 1 \end{pmatrix} \quad (6)$$

is also a solution, provided that  $\partial_\pm \tilde{h} = \pm \frac{r}{\Delta^2} \partial_\pm h$ , where  $\sqrt{2}x^\pm = t \pm r$  [15]. This symmetry, which can be generalized to  $n > 2$  case (at least for some special wave polarizations, see section 3), implies that  $d\tilde{s}^2$  can have the asymptotic behavior of a flat metric in cylindrical coordinates, i.e. when  $r \rightarrow \infty$  then

$$d\tilde{s}^2 \sim -e^\gamma(dt^2 - dr^2) + r^2 d\chi_1^2 + d\chi_2^2 + d\chi_3^2 + \dots + d\chi_n^2, \quad (7)$$

where  $\gamma$  is a constant. Existence of this duality symmetry relates the original solution to cylindrically symmetric solution, where coordinate  $\chi_1$  is the angle of rotation around the axis  $\chi_2$ . We then require that the original solution has the asymptotics

$$ds^2 \sim -e^\gamma(dt^2 - dr^2) + d\chi_1^2 + d\chi_2^2 + d\chi_3^2 + \dots + d\chi_n^2. \quad (8)$$

In order to find a free-field formulation for arbitrary  $n$ , we need to generalize the  $n = 2$  parametrization of the  $\Lambda$  matrix [10]

$$\Lambda = \begin{pmatrix} e^f + h^2 e^{-f} & e^{-f} h \\ e^{-f} h & e^{-f} \end{pmatrix}. \quad (9)$$

Note that for  $n \geq 2$ ,  $\Lambda$  can be written as

$$\Lambda = \begin{pmatrix} N + h^T g h & g h \\ g h & g \end{pmatrix}, \quad (10)$$

where  $g$  is a symmetric  $(n - 1) \times (n - 1)$  matrix,  $h$  is a  $(n - 1)$ -dimensional vector (column) and  $N = (\det g)^{-1}$ . Given a  $\Lambda_{n-1}$ , the parametrization (10) implies that  $\Lambda_n$  can be written as

$$\Lambda_n = \begin{pmatrix} e^{(n-1)f_{n-1}} + e^{-f_{n-1}} h_{n-1}^T \Lambda_{n-1} h_{n-1} & e^{-f_{n-1}} \Lambda_{n-1} h_{n-1} \\ e^{-f_{n-1}} \Lambda_{n-1} h_{n-1} & e^{-f_{n-1}} \Lambda_{n-1} \end{pmatrix}. \quad (11)$$

The recursive relation (11) gives the parametrization of  $\Lambda_n$  in terms of  $f_k$  and  $h_k$  fields, where  $k = 1, 2, \dots, n - 1$ . Note that there is a more general parametrization given by setting  $e^{f_{n-1}} = \Delta_{n-1}$  in (11), where  $\Delta_k$  can be both positive and negative (for  $n = 2$  this is the parametrization (5)). However, the asymptotics (8) requires that all  $\Delta_k$  be positive, and hence we use the parametrization (11).

From (10) it follows that

$$\Lambda^{-1} = \begin{pmatrix} N^{-1} & -N^{-1} h \\ -N^{-1} h & g^{-1} + N^{-1} h h^T \end{pmatrix}, \quad (12)$$

so that

$$\Lambda^{-1} d\Lambda = \begin{pmatrix} N^{-1}(dN + h^T g dh) & N^{-1} g dh \\ dh + (g^{-1} dg)h - N^{-1}(dN + h^T g dh)h & g^{-1} dg - N^{-1}(g dh)h^T \end{pmatrix}. \quad (13)$$

By using (13) we obtain

$$tr(\Lambda^{-1} \partial_\mu \Lambda \Lambda^{-1} \partial_\nu \Lambda) = N^{-2} \partial_\mu N \partial_\nu N + 2N^{-1} \partial_\mu h^T g \partial_\nu h + tr(g^{-1} \partial_\mu g g^{-1} \partial_\nu g). \quad (14)$$

By using the recursive relation (14) and the parametrization formula (11) it is easy to obtain

$$tr(\Lambda^{-1} \partial_\mu \Lambda \Lambda^{-1} \partial_\nu \Lambda) = \sum_{k=1}^{n-1} [(k^2 + k) \partial_\mu f_k \partial_\nu f_k + 2e^{-(k+1)f_k} (\partial_\mu h_k)^T \Lambda_k \partial_\nu h_k]. \quad (15)$$

The action (3) can be now written as

$$S = \int d^2 x \sqrt{-g} \varphi \left[ R - \frac{1}{4} \sum_{k=1}^{n-1} [(k^2 + k)(\nabla f_k)^2 + 2e^{-(k+1)f_k} (\nabla h_k)^T \Lambda_k \nabla h_k] \right], \quad (16)$$

which is a natural generalization of the  $n = 2$  action of [10].

The two-dimensional diffeomorphism invariance of this action implies that its Hamiltonian form is given by

$$S = \int dt dr \left[ \pi_\rho \dot{\rho} + \pi_\varphi \dot{\varphi} + \sum_{k=1}^{n-1} \left( p_k \dot{f}_k + \pi_k^T \dot{h}_k \right) - N_0 G_0 - N_1 G_1 \right], \quad (17)$$

where  $\rho = \log g_{11}$ , and  $G_0$  and  $G_1$  are the constraints given by

$$\begin{aligned} G_0 &= -\pi_\rho \pi_\varphi - \rho' \varphi' + 2\varphi'' \\ &+ \sum_{k=1}^{n-1} \left[ \frac{p_k^2}{(k^2 + k)\varphi} + \frac{1}{4}(k^2 + k)\varphi(f'_k)^2 \right] \\ &+ \sum_{k=1}^{n-1} \left[ \frac{e^{(k+1)f_k}}{2\phi} \pi_k^T \Lambda_k^{-1} \pi_k + \frac{1}{2}\varphi e^{-(k+1)f_k} (h'_k)^T \Lambda_k (h'_k) \right] \\ G_1 &= \pi_\rho \rho' - 2\pi'_\rho + \pi_\varphi \varphi' + \sum_{k=1}^{n-1} \left[ p_k f'_k + \pi_k^T h'_k \right] . \end{aligned} \quad (18)$$

Dots represent the  $t$  derivatives and primes represent the  $r$  derivatives, and the Lagrange multipliers  $N_0$  and  $N_1$  are related to the components of the two-dimensional metric as

$$g_{00} = -N_0^2 + g_{11}N_1^2 \quad , \quad g_{01} = g_{11}N_1 \quad . \quad (19)$$

The Poisson bracket algebra of the constraints is isomorphic to the two-dimensional diffeomorphism algebra, and hence the constraints generate the two-dimensional infinitesimal diffeomorphism transformations. The algebra of constraints splits into a direct sum of two one-dimensional diffeomorphism algebras via  $C_\pm = \frac{1}{2}(G_0 \pm G_1)$ .

### 3 Equations of motion

We will study the dynamics of the action (16) in the conformal gauge for the two-dimensional metric

$$g_{++} = g_{--} = 0 \quad , \quad g_{+-} = -e^\rho \quad , \quad (20)$$

where  $\rho$  is the two-dimensional conformal factor. This gauge corresponds to  $N_0 = 1$  and  $N_1 = 0$  in the Hamiltonian formulation, and  $g_{11} = e^\rho$ .

Let us write the action (3) in the conformal gauge

$$\begin{aligned} S_c &= \int dt dr [2\varphi \partial_+ \partial_- \rho \\ &+ \frac{1}{2}\varphi \sum_{k=1}^{n-1} [(k^2 + k)\partial_+ f_k \partial_- f_k + 2e^{-(k+1)f_k} (\partial_+ h_k)^T \Lambda_k \partial_- h_k] ] . \end{aligned} \quad (21)$$

The variation of  $S_c$  with respect to  $\rho$  gives

$$\partial_+ \partial_- \varphi = 0 , \quad (22)$$

which implies that  $\varphi$  is a free field. This equation is used to completely fix the gauge, and in our case one takes

$$\varphi = \frac{1}{\sqrt{2}}(x^+ - x^-) = r . \quad (23)$$

However, in order to display the free-field structure, we will take the general solution

$$\varphi = A_+(x^+) + A_-(x^-) , \quad (24)$$

where  $A_+(x^+)$  and  $A_-(x^-)$  are monotonic increasing and decreasing functions respectively, which go as  $\frac{1}{\sqrt{2}}x^+$  and  $-\frac{1}{\sqrt{2}}x^-$  when  $x^+ \rightarrow \infty$  and  $x^- \rightarrow -\infty$  respectively.

Variation of  $S_c$  with respect to  $\varphi$  gives

$$2\partial_+ \partial_- \rho + \frac{1}{2} \sum_{k=1}^{n-1} [(k^2 + k)\partial_+ f_k \partial_- f_k + 2e^{-(k+1)f_k} (\partial_+ h_k)^T \Lambda_k \partial_- h_k] = 0 , \quad (25)$$

which can be solved as

$$\begin{aligned} \rho &= a_+(x^+) + a_-(x^-) \\ &+ \frac{1}{4} \int_{x^+}^{\infty} dy^+ \int_{-\infty}^{x^-} dy^- \sum_{k=1}^{n-1} (k^2 + k) \partial_+ f_k \partial_- f_k \\ &+ \frac{1}{2} \int_{x^+}^{\infty} dy^+ \int_{-\infty}^{x^-} dy^- \sum_{k=1}^{n-1} e^{-(k+1)f_k} (\partial_+ h_k)^T \Lambda_k \partial_- h_k , \end{aligned} \quad (26)$$

where  $a_{\pm}$  are two arbitrary chiral functions.

The non-trivial dynamics is contained in the Ernst equation (4), which is obtained by varying the action with respect to  $\Lambda$ , or equivalently by varying  $S_c$  with respect to  $f_k$

$$\begin{aligned} k[\partial_+(\varphi \partial_- f_k) + \partial_-(\varphi \partial_+ f_k)] + 2\varphi e^{-(k+1)f_k} (\partial_+ h_k)^T \Lambda_k \partial_- h_k \\ - \frac{2}{k+1} \varphi \sum_{j=k+1}^{n-1} e^{-(j+1)f_j} (\partial_+ h_j)^T \frac{\partial \Lambda_j}{\partial f_k} \partial_- h_j = 0 , \end{aligned} \quad (27)$$

and  $h_k$

$$\begin{aligned} \partial_+(\varphi e^{-(k+1)f_k} \Lambda_k \partial_- h_k) + \partial_-(\varphi e^{-(k+1)f_k} \Lambda_k \partial_+ h_k) \\ - \varphi \sum_{j=k+1}^{n-1} e^{-(j+1)f_j} (\partial_+ h_j)^T \frac{\partial \Lambda_j}{\partial h_k} \partial_- h_j = 0 . \end{aligned} \quad (28)$$

In order to determine the chiral functions  $a_{\pm}$ , one needs the constraint equations, which cannot be obtained from  $S_c$ , but instead one must vary the full action with respect to  $g_{++}$  and  $g_{--}$  and then impose the conformal gauge conditions. In this way one obtains

$$\begin{aligned} C_{\pm} &= \partial_{\pm}^2 \varphi - \partial_{\pm} \varphi \partial_{\pm} \rho \\ &+ \frac{1}{4} \varphi \sum_{k=1}^{n-1} [(k^2 + k)(\partial_{\pm} f_k)^2 + 2e^{-(k+1)f_k} (\partial_{\pm} h_k)^T \Lambda_k \partial_{\pm} h_k] = 0 \quad . \end{aligned} \quad (29)$$

Imposing the constraint equations (29) also requires fixing the functions  $A_{\pm}$ , since in this way one completely fixes the two-dimensional diffeomorphism invariance. The action  $S_c$  without the constraints (29) is invariant under two-dimensional conformal transformations, which are two-dimensional diffeomorphism which preserve the conformal gauge.

The  $n = 2$  duality symmetry can be generalized to arbitrary  $n$  for special polarizations. In the collinear polarization case, which is also referred to as the Abelian case, we have  $h_k = 0$ , and a dual solution is given by

$$\tilde{f}_{n-1} = -f_{n-1} + \frac{2}{n} \log r \quad , \quad \tilde{f}_k = f_k \quad , \quad k \leq n-2 \quad . \quad (30)$$

For the case when only  $f_{n-1}$  and  $h_{n-1}$  are non-zero, a dual solution is given by

$$\tilde{f}_{n-1} = -f_{n-1} + \frac{2}{n} \log r \quad , \quad \partial_{\pm} \tilde{h}_{n-1} = \pm r e^{-nf_{n-1}} \partial_{\pm} h_{n-1} \quad . \quad (31)$$

These solutions have the asymptotics (7), and hence they are manifestly cylindrically symmetric. In general case we do not know the form of the dual solution with the cylindrical asymptotics, but independently of that the asymptotics (8) must be satisfied, which is equivalent to requiring that

$$\lim_{r \rightarrow \infty} f_k = 0 \quad , \quad \lim_{r \rightarrow \infty} h_k = 0 \quad . \quad (32)$$

## 4 Free fields

The free-field construction of [10] is based on the fact that the two-dimensional diffeomorphism algebra of constraints admits representations quadratic in canonical variables

$$G_0 = \frac{1}{2} (\eta^{ij} P_i P_j + \eta_{ij} Q^{i\prime} Q^{j\prime}) \quad , \quad G_1 = P_i Q^{i\prime} \quad . \quad (33)$$

Where  $i, j = 1, \dots, m$  and  $\eta_{ij}$  is a flat Minkowskian metric. Note that the quadratic representations of diffeomorphism constraints in higher dimensions are not possible, since then the constraint algebra has structure functions which are not constants. The

representation (33) implies that  $Q^i$  are free fields and since (3) is an integrable two-dimensional theory there is a possibility to find a canonical transformation from the initial canonical variables to the free-field canonical variables  $(P_i, Q^i)$ , so that the number of free fields would be  $m = 1 + \frac{1}{2}n(n+1)$ .

In order to show this, we go to the conformal gauge and insert the solutions for  $\rho$  and  $\varphi$  into the constraints. We obtain

$$\begin{aligned} C_{\pm} &= \partial_{\pm}^2 A_{\pm} - \partial_{\pm} A_{\pm} \partial_{\pm} a_{\pm} \\ &\quad + \frac{1}{4} \partial_{\pm} A_{\pm} \int_{\mp\infty}^{x^{\mp}} dy^{\mp} \sum_k [(k^2 + k) \partial_+ f_k \partial_- f_k + 2e^{-(k+1)f_k} (\partial_+ h_k)^T \Lambda_k \partial_- h_k] \\ &\quad + \frac{1}{4} \varphi \sum_k [(k^2 + k) (\partial_{\pm} f_k)^2 + 2e^{-(k+1)f_k} (\partial_{\pm} h_k)^T \Lambda_k \partial_{\pm} h_k] = 0 . \end{aligned} \quad (34)$$

From  $\partial_{\mp} C_{\pm} = 0$  it follows that  $\partial_{\mp} P_{\pm} = 0$ , where

$$\begin{aligned} P_{\pm} &= \frac{1}{4} \partial_{\pm} A_{\pm} \int_{\mp\infty}^{x^{\mp}} dy^{\mp} \sum_k [(k^2 + k) \partial_+ f_k \partial_- f_k + 2e^{-(k+1)f_k} (\partial_+ h_k)^T \Lambda_k \partial_- h_k] \\ &\quad + \frac{1}{4} \varphi \sum_k [(k^2 + k) (\partial_{\pm} f_k)^2 + 2e^{-(k+1)f_k} (\partial_{\pm} h_k)^T \Lambda_k \partial_{\pm} h_k] . \end{aligned} \quad (35)$$

Since  $P_{\pm}$  are independent of  $x^{\mp}$ ,  $P_{\pm}$  can be evaluated by taking the limits  $x^{\mp} \rightarrow \mp\infty$ , since then the integral terms in (34) vanish. This gives

$$P_{\pm} = \lim_{x^{\mp} \rightarrow \mp\infty} \frac{1}{4} \varphi \sum_k [(k^2 + k) (\partial_{\pm} f_k)^2 + 2e^{-(k+1)f_k} (\partial_{\pm} h_k)^T \Lambda_k \partial_{\pm} h_k] . \quad (36)$$

We perform a change of variables

$$\sqrt{\frac{k^2 + k}{2}} f_k \sqrt{\varphi} = \tilde{F}_k , \quad h_k \sqrt{\varphi} = \tilde{H}_k \quad (37)$$

where  $\tilde{F}, \tilde{H}$  and their derivatives are bounded in the limit  $\varphi \rightarrow \infty$ . This is in agreement with the boundary conditions (32), and the equations (27) and (28) can be written as

$$\partial_+ \partial_- \tilde{F}_k + O\left(\frac{1}{\sqrt{\varphi}}\right) = 0 , \quad (38)$$

$$\partial_+ \partial_- \tilde{H}_k + O\left(\frac{1}{\sqrt{\varphi}}\right) = 0 . \quad (39)$$

Therefore when  $\varphi \rightarrow \infty$ , one has

$$\sqrt{\frac{k^2 + k}{2}} f_k \sim \frac{F_k}{\sqrt{\varphi}} , \quad h_k \sim \frac{H_k}{\sqrt{\varphi}} , \quad (40)$$

where  $F_k$  and  $H_k$  are bounded free fields with bounded derivatives.

When the asymptotics (40) is combined with the result (36), we obtain

$$P_{\pm} = \frac{1}{2} \sum_k [(\partial_{\pm} F_k)^2 + (\partial_{\pm} H_k)^T \partial_{\pm} H_k] . \quad (41)$$

If one defines

$$X^{\pm} = A_{\pm}, \quad \Pi_{\pm} = -\partial_{\pm} a_{\pm} + \frac{\partial_{\pm}^2 A_{\pm}}{\partial_{\pm} A_{\pm}}, \quad (42)$$

the constraints (29) take a free-field form

$$C_{\pm} = \Pi_{\pm} \partial_{\pm} X^{\pm} + \frac{1}{2} \sum_k [(\partial_{\pm} F_k)^2 + (\partial_{\pm} H_k)^T \partial_{\pm} H_k] . \quad (43)$$

## 5 Canonical transformation

We have found a map from  $\rho, \varphi, f, h$  variables to free-field variables  $\Pi_{\pm}, X^{\pm}, F_k, H_k$  defined by (42) and (40), but it is not clear whether this map represents a canonical transformation. Namely, although the constraints take a free-field form (43), one has to show that

$$\begin{aligned} & \int_{t=const.} dr \left[ \pi_{\rho} \dot{\rho} + \pi_{\varphi} \dot{\varphi} + \sum_k \left( \pi_k \dot{f}_k + (\pi_k)^T \dot{h}_k \right) \right] \\ &= \int_{t=const.} dr \left[ \Pi_+ \dot{X}^+ + \Pi_- \dot{X}^- + \sum_k \left( P_k \dot{F}_k + \Pi_k (\dot{H}_k)^T \right) \right] , \end{aligned} \quad (44)$$

up to total time derivative. This can be shown by examining the pre-symplectic form on the unconstrained phase space given by  $\rho, \varphi, f, h$  and their canonically conjugate momenta. This phase space is the same as the space of solutions corresponding to the action  $S_c$  (21), i.e. solutions in the conformal gauge for which the constraint equations (29) are not imposed, so that  $A_{\pm}$  and  $a_{\pm}$  are not fixed.

The unconstrained pre-symplectic one-form is defined as

$$\begin{aligned} \Omega &= \int_{t=const.} dr \left[ \pi_{\rho} \delta \rho + \pi_{\varphi} \delta \varphi + \sum_k [\pi_k \delta f_k + (\pi_k)^T \delta h_k] \right] \\ &= \int_{t=const.} dr \left[ \frac{\partial L}{\partial \dot{\rho}} \delta \rho + \frac{\partial L}{\partial \dot{\varphi}} \delta \varphi + \sum_k \left[ \frac{\partial L}{\partial \dot{f}_k} \delta f_k + \left( \frac{\partial L}{\partial h_k} \right)^T \delta h_k \right] \right] \end{aligned} \quad (45)$$

where  $L$  is the Lagrangean density of  $S_c$ . This can be rewritten in the covariant form

$$\Omega = \int_{\Sigma} d\sigma_{\mu} \frac{\partial L}{\partial (\partial_{\mu} \Phi_a)} \delta \Phi_a = \int_{\Sigma} d\sigma_{\mu} j^{\mu} , \quad (46)$$

where  $\delta$  stands for the exterior derivative on the space of solutions of the equations of motion,  $\Sigma$  is a spacelike hypersurface and  $j^\mu$  is the symplectic current one-form [16]. Because the symplectic current is conserved  $\partial_\mu j^\mu = 0$ , the definition (46) is independent of the choice of the hypersurface  $\Sigma$ .

By integrating  $\partial_\mu j^\mu$  over the regions bounded by  $t = \text{const.}$  and  $x^\pm = \pm\infty$  hypersurfaces, one can derive

$$\Omega = \frac{1}{2} \int_{x^-=-\infty} dx^+ j^- + \frac{1}{2} \int_{x^+=+\infty} dx^- j^+. \quad (47)$$

The light-cone components of the one-form current  $j^\mu$  can be calculated from  $S_c$

$$j^+ = \partial_- \varphi \delta\rho + \partial_- \rho \delta\varphi - \frac{1}{2} \varphi \sum_k [(k^2 + k) \partial_- f_k \delta f_k + 2e^{-(k+1)f_k} (\partial_- h_k)^T \Lambda_k \delta h_k], \quad (48)$$

$$j^- = \partial_+ \varphi \delta\rho + \partial_+ \rho \delta\varphi - \frac{1}{2} \varphi \sum_k [(k^2 + k) \partial_+ f_k \delta f_k + 2e^{-(k+1)f_k} (\partial_+ h_k)^T \Lambda_k \delta h_k]. \quad (49)$$

By taking into account the asymptotic behavior of  $\rho, \varphi, f_k$  and  $h_k$  for  $x^\pm \rightarrow \pm\infty$  and equation (42), it is easy to see that the symplectic two-form  $\omega = \delta\Omega$  is given by

$$\begin{aligned} \omega &= \frac{1}{2} \int_{x^-=-\infty} dx^+ \left[ \delta X^+ \wedge \delta \Pi_+ + \sum_k (\delta F_{k+} \wedge \delta \partial_+ F_k + \delta H_{+k}^T \wedge \delta \partial_+ H_k) \right] \\ &+ \frac{1}{2} \int_{x^+=\infty} dx^- \left[ \delta X^- \wedge \delta \Pi_- + \sum_k (\delta F_{k-} \wedge \delta \partial_- F_k + \delta H_{k-}^T \wedge \delta \partial_- H_k) \right]. \end{aligned} \quad (50)$$

By using (47) we get

$$\omega = \int_{t=\text{const.}} dr \left[ \delta X^+ \wedge \Pi_+ + \delta X^- \wedge \delta \Pi_- + \sum_k (\delta F_k \wedge \delta \dot{F}_k + \delta H_k^T \wedge \delta \dot{H}_k) \right], \quad (51)$$

where  $F_\pm(x^\pm), H_\pm(x^\pm)$  are the chiral parts of the free fields  $F$  and  $H$ , ( $F = F_+ + F_-, H = H_+ + H_-$ ). From (51) it follows that

$$\begin{aligned} \Omega &= \int_{t=\text{const.}} dr \left[ \pi_\rho \delta\rho + \pi_\varphi \delta\varphi + \sum_k (\pi_k \delta f_k + \pi_k^T \delta h_k) \right] \\ &= \int_{t=\text{const.}} dr \left[ \Pi_+ \delta X^+ + \Pi_- \delta X^- + \sum_k (\dot{F}_k \delta F_k + (\dot{H}_k)^T \delta H_k) \right], \end{aligned} \quad (52)$$

which implies

$$\begin{aligned} &\int_{t=\text{const.}} dr \left[ \pi_\rho \dot{\rho} + \pi_\varphi \dot{\varphi} + \sum_k (\pi_k \dot{f}_k + \pi_k^T \dot{h}_k) \right] \\ &= \int_{t=\text{const.}} dr \left[ \Pi_+ \dot{X}^+ + \Pi_- \dot{X}^- + \sum_k (\dot{F}_k \dot{F}_k + (\dot{H}_k)^T \dot{H}_k) \right], \end{aligned} \quad (53)$$

up to a total time derivative. From (53) and (17) it follows that

$$S_c = \int dt dr \left[ \Pi_+ \dot{X}^+ + \Pi_- \dot{X}^- + \sum_k \left( \dot{F}_k \dot{F}_k + (\dot{H}_k)^T \dot{H}_k \right) - (C_+ + C_-) \right], \quad (54)$$

where  $C_{\pm}$  are given by (29). From (54) it follows that  $P_k = \dot{F}_k$  and  $\Pi_k = \dot{H}_k$ , and hence (53) gives (44). Therefore we have a canonical transformation.

In terms of the canonical variables, (43) can be written as

$$C_{\pm} = \pm \Pi_{\pm} X'^{\pm} + \frac{1}{4} \sum_k \left[ (P_k \pm F'_k)^2 + (\Pi_k \pm H'_k)^T (\Pi_k \pm H'_k) \right]. \quad (55)$$

By performing a canonical transformation

$$2X^{\pm'} = \mp(\Pi_1 - \Pi_0) - X^{0'} - X^{1'} , \quad 2\Pi_{\pm} = -\Pi_0 - \Pi_1 \mp (X^{1'} - X^{0'}) \quad (56)$$

the constraints take the form (33).

## 6 Free-field expansions

Although we have shown that a free-field formulation exists, what one really needs are more explicit expressions for  $f_k$  and  $h_k$  fields in terms of the free fields  $F_k$  and  $H_k$  than the ones given by the asymptotic relations (40).

In the Abelian case the asymptotics (40) is explicitly realized because the exact solution for the fields  $f_k$  is given by

$$f_k = \int_{-\infty}^{\infty} d\lambda J_0(\lambda r) [A_{+k}(\lambda) e^{i\lambda t} + A_{-k}(\lambda) e^{-i\lambda t}], \quad (57)$$

where  $J_0$  is the Bessel function and  $A_{\pm k}(\lambda)$  are arbitrary coefficients. When  $r \rightarrow \infty$ , this behaves as

$$\begin{aligned} f_k \sim & \frac{1}{\sqrt{2r}} \int_{-\infty}^{\infty} d\lambda (\pi |\lambda|)^{-\frac{1}{2}} \left[ A_{+k}(\lambda) e^{i\lambda x^+} e^{-i\frac{\pi}{4}} + A_{-k}(\lambda) e^{-i\lambda x^+} e^{i\frac{\pi}{4}} \right. \\ & \left. + A_{+k}(\lambda) e^{i\lambda x^-} e^{-i\frac{\pi}{4}} + A_{-k}(\lambda) e^{-i\lambda x^-} e^{i\frac{\pi}{4}} \right] . \end{aligned} \quad (58)$$

From (40) it follows that

$$\begin{aligned} F_k = & \int_{-\infty}^{\infty} d\lambda (\pi |\lambda|)^{-\frac{1}{2}} \left[ A_{+k}(\lambda) e^{i\lambda x^+} e^{-i\frac{\pi}{4}} + A_{-k}(\lambda) e^{-i\lambda x^+} e^{i\frac{\pi}{4}} \right. \\ & \left. + A_{+k}(\lambda) e^{i\lambda x^-} e^{-i\frac{\pi}{4}} + A_{-k}(\lambda) e^{-i\lambda x^-} e^{i\frac{\pi}{4}} \right] , \end{aligned} \quad (59)$$

and therefore  $F$  is a bounded free field and  $\partial_+ F, \partial_- F, \dots$  are also bounded. The relations (57) and (59) give an exact relationship between  $f_k$  and  $F_k$ , and one can obtain from them an exact expression for  $f_k$  in terms of  $F_k$ .

In the non-abelian case the explicit form of the solutions is not known. However, in  $D = 4$  case there is an asymptotic series expansion of  $f$  and  $h$  in terms of  $F, H$  and  $\varphi$  which could in principle give a non-abelian generalization of the expression (57) [10]. The idea is to write  $f$  and  $h$  as an asymptotic series expansions

$$f = \frac{F}{\sqrt{\varphi}} + \sum_{i=1}^{\infty} \frac{F_{(i)}}{(\sqrt{\varphi})^{i+1}} , \quad (60)$$

and

$$h = \frac{H}{\sqrt{\varphi}} + \sum_{i=1}^{\infty} \frac{H_{(i)}}{(\sqrt{\varphi})^{i+1}} , \quad (61)$$

where  $F_{(i)}$  and  $H_{(i)}$  are functionals of  $F$  and  $H$ . The form of these functionals can be determined from the Ernst equations, and this can be done explicitly because one obtains the recurrence relations

$$\begin{aligned} \partial_+ \partial_- F_{(i)} &= -\partial_r F_{(i-2)} - \frac{(i-3)^2}{16} F_{(i-4)} + \mathcal{F}_{(i-1)} \\ \partial_+ \partial_- H_{(i)} &= -\partial_r H_{(i-2)} - \frac{(i-3)^2}{16} H_{(i-4)} + \mathcal{H}_{(i-1)} \end{aligned} , \quad (62)$$

where

$$\mathcal{F}_{(i)} = \sum_{j+\dots+k+l+m=i} a_{lm}^{j\dots k} F_{(j)} \cdots F_{(k)} D_+ H_{(l)} D_- H_{(m)} \quad (63)$$

and

$$\mathcal{H}_{(i)} = \sum_{j+k=i} [D_+ F_{(j)} D_- H_{(k)} + D_- F_{(j)} D_+ H_{(k)}] . \quad (64)$$

We have introduced new derivatives  $D_{\pm} X_{(i)} = \partial_{\pm} X_{(i)} \mp \frac{(i-1)}{4} X_{(i-2)}$  and  $a_{lm}^{j\dots k}$  are numerical coefficients, while  $F_{(0)} = F$ ,  $H_{(0)} = H$  and  $F_{(i)} = H_{(i)} = 0$  for  $i < 0$ . Hence all the higher-order  $F$  and  $H$  are determined from the zero-order ones. In this way one obtains

$$F_{(1)} = -\frac{1}{2} H^2 , \quad H_{(1)} = FH \quad (65)$$

and so on.

The same pyramid structure as (62) is preserved in the  $D > 4$  case, and the difference is that the polynomial function  $\mathcal{F}_{(i)}$  depends on higher than two powers of  $H$

$$\mathcal{F}_{(i)} = \sum_{j+\dots+k+l+\dots+m+n+p=i} a_{l\dots mnp}^{j\dots k} F_{(j)} \cdots F_{(k)} H_{(l)} \cdots H_{(m)} D_+ H_{(n)} D_- H_{(p)} , \quad (66)$$

while  $\mathcal{H}_{(i)}$  becomes a polynomial of order  $i + 2$

$$\mathcal{H}_{(i)} = \sum_{j+\dots+k+l+\dots+m+n+p=i} b_{l\dots mnp}^{j\dots k} F_{(j)} \cdots F_{(k)} H_{(l)} \cdots H_{(l)} D_+ F_{(n)} D_- H_{(p)}$$

$$\begin{aligned}
& + \sum_{j+\dots+k+l+\dots+m+n+p=i} b_{l\dots mnp}^{j\dots k} F_{(j)} \cdots F_{(k)} H_{(l)} \cdots H_{(l)} D_- F_{(n)} D_+ H_{(p)} \\
& + \sum_{j+\dots+k+l+\dots+m+n+p=i} c_{l\dots mnp}^{j\dots k} F_{(j)} \cdots F_{(k)} H_{(l)} \cdots H_{(m)} D_+ H_{(n)} D_- H_{(p)} \quad (67)
\end{aligned}$$

where  $a, b$  and  $c$  are numerical coefficients. Note that dependence on  $\partial F$  and  $\partial H$  terms is always a quadratic polynomial, because the Ernst equations have only two derivatives.

This structure of  $\mathcal{F}$  and  $\mathcal{H}$  polynomials in  $D > 4$  case comes from the fact that  $\Lambda_n$  matrix is non-trivial. The matrix elements of  $\Lambda_n$  are polynomials in  $e^{f_j}$  and  $h_l$ , which follows from the parametrization (11). Furthermore, the expansions (60) and (61) imply

$$\Lambda_k = \sum_{j=0}^{\infty} \Lambda_k^{(j)} \varphi^{-j/2} \quad (68)$$

where  $\Lambda_k^0$  is a constant matrix, and  $\Lambda_k^{(j)}$  for  $j > 0$  are polynomials in  $F_{(l)}$  and  $H_{(m)}$  where  $l, m \leq j$ . The same is valid for  $\partial \Lambda_k / \partial f_j$  and  $\partial \Lambda_k / \partial h_j$ , so that the Ernst equations give (62) with (67) and (66). An analog of (65) is

$$F_k^{(1)} = -\frac{1}{2k} \left[ H_k^T H_k - \frac{1}{k+1} \sum_{j=k+1}^{n-1} H_j^T H_j \right] , \quad H_k^{(1)} = F_k H_k . \quad (69)$$

When  $H = 0$ , one recovers in this way the large  $r$  asymptotic expansion of the Bessel function, which is the exact solution in the Abelian case.

## 7 Conclusions

We have shown that the  $D$ -dimensional cylindrically symmetric general relativity on spacetimes with non-compact spatial sections can be mapped via canonical transformation into a two-dimensional diffeomorphism invariant theory of  $1 + \frac{1}{2}(D-1)(D-2)$  free fields. In the case when the spatial section is compact, which corresponds to cosmological models, the situation with a free-field realization is more complicated. In this case the simplifications due to  $r \rightarrow \infty$  asymptotics can not be used, since  $\varphi = t$  is a consistent gauge choice. It is clear that for  $t \rightarrow \pm\infty$  one can have asymptotic free-field solutions, but if one wants to prove the existence of free-field variables for all times one must find the exact expressions for the free fields in terms of the  $D$ -dimensional metric variables. This can be done in the  $D = 4$  case [10]

$$(\partial_{\pm} F)^2 = \varphi (\partial_{\pm} f)^2 + \partial_{\pm} \varphi \int_{x_0^{\mp}}^{x^{\mp}} dy^{\mp} \partial_+ f \partial_- f + 2 \int_{x_0^{\mp}}^{x^{\mp}} dy^{\mp} \varphi e^{-2f} \partial_{\pm} f \partial_+ h \partial_- h, \quad (70)$$

$$(\partial_{\pm} H)^2 = \varphi e^{-2f} (\partial_{\pm} h)^2 + \partial_{\pm} \varphi \int_{x_0^{\mp}}^{x^{\mp}} dy^{\mp} e^{-2f} \partial_+ h \partial_- h - 2 \int_{x_0^{\mp}}^{x^{\mp}} dy^{\mp} \varphi e^{-2f} \partial_{\pm} f \partial_+ h \partial_- h. \quad (71)$$

These expressions are independent of spatial topology, since they follow from the equations of motion, which are local. In order to see whether the transformations (70) and (71) define a canonical transformation, one needs to examine the pre-symplectic form as in the section 5, but without using the  $\varphi \rightarrow \infty$  asymptotics. Proving this and finding a  $D$ -dimensional generalization of (70) and (71) is an open problem. Therefore, although free-fields exist in the compact case, it is not clear whether a canonical transformation to free-fields exists. However, the asymptotic expansions (60) and (61) can be still used for obtaining the late-time solutions.

The quantization of cylindrical gravitational waves in terms of the free-field variables is considerably simpler than if one uses the observables obtained from the BZM approach [18]. The BZM observables form a non-linear Yangian algebra, whose Hilbert space representations is difficult to find [18]. In the free-field approach the observables are given by the "gravitationally dressed" Fourier modes [17], defined by

$$F_k = \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{d\lambda}{|\lambda|} [e^{i\lambda_\mu X^\mu} A_k(\lambda) + e^{-i\lambda_\mu X^\mu} A_k^*(\lambda)] , \quad (72)$$

$$H_k = \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{d\lambda}{|\lambda|} [e^{i\lambda^\mu X_\mu} B_k(\lambda) + e^{-i\lambda^\mu X_\mu} B_k^*(\lambda)] , \quad (73)$$

where  $\lambda^\mu X_\mu = \lambda_+ X^+ + \lambda_- X^-$  and  $\lambda_\pm = \frac{1}{2}(\lambda \mp |\lambda|)$ . Upon quantization,  $A(\lambda)$  and  $B(\lambda)$  become creation and annihilation operators, and the corresponding Fock space is the Hilbert space of the quantum theory. The only subtlety in this construction is that the algebra of the quantum constraints  $C_\pm$  has an anomaly, which can be canceled by using modified quantum constraints [8]

$$\tilde{C}_\pm = C_\pm + \frac{c}{48\pi} \left[ \frac{X^{\pm'''}}{X^{\pm'}} - \left( \frac{X^{\pm''}}{X^{\pm'}} \right)^2 \right] , \quad (74)$$

where  $c$  is equal to the number of physical scalar fields, so that  $c = \frac{1}{2}(D-1)(D-2) - 1$ . A less straightforward task will be finding the expectation values of the metric variables, since they become complicated functionals of the free fields. However, the expansions (60) and (61) become suitable for such a task.

The Fourier modes  $A(\lambda)$  and  $B(\lambda)$  constitute a complete set of observables, and therefore the BZM observables could be in principle expressed in terms of them. Since the BZM observables form a Yangian  $sl(n, R)$  algebra, this implies that there should be a free-field representation of these non-linear algebras. Also note that for  $D = 4$  one can construct an affine  $sl(2, R)$  algebra from the BZM observables, and this algebra generates the Geroch group which is the dynamical symmetry of the theory [19]. This symmetry can be easily seen in the free-field approach, since one can construct the generators of an affine  $sl(2, R)$  algebra from  $A(\lambda)$  and  $B(\lambda)$  via the Wakimoto construction [20]. Furthermore, for general  $D$  there exists a generalized Wakimoto construction [21, 22], which gives the generators of affine  $sl(n, R)$  algebra in terms of  $\frac{1}{2}n(n+1) - 1$  creation and annihilation operators, which is exactly the number

of  $A_k(\lambda)$  and  $B_k(\lambda)$ . This algebra will generate a dynamical symmetry group in the  $D$ -dimensional case.

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